1 First-quantized particle production

Let's do particle production in first-quantized form so as to be able to generalize to strings. For the time-dependent problem

$$(\partial_t^2 + m^2(t))\psi(t) = 0\,, \tag{1}$$

we want

$$\frac{\langle \text{out}|a_{\text{out}}^2|\text{in}\rangle}{\langle \text{out}|\text{in}\rangle}\,, \tag{2}$$

which we get from the positive frequency part of

$$\frac{\langle \text{out}|\phi(t_1)\phi(t_2)|\text{in}\rangle}{\langle \text{out}|\text{in}\rangle}\,. \tag{3}$$

The latter is equal to

$$-i\left[\partial_t^2 + m^2(t) - i\epsilon\right]^{-1}(t_1,t_2) = \int_0^\infty d\tau \int \mathcal{D}t\Big|_{t_1}^{t_2} \exp\left\{-\frac{i}{2}\int_0^\tau d\tau'\,(\dot{t}^2 + m^2(t))\right\}\,. \tag{4}$$

[Change of notation, I use τ and dot for worldline time and derivative, and μ for tension later. Derivatives with respect to real time are explicitly ∂_t.] At a saddle point (in both t and τ), the configuration satisfies the constraint

$$\dot{t}^2 = m^2(t)\,, \tag{5}$$

and the action becomes

$$S \to \mp \int_0^\tau d\tau'\, m(t)\dot{t} = \mp \int_{t_1}^{t_2} dt\, m(t)\,. \tag{6}$$

For the LL problem,

$$m(t) = (a^2 + b^2 t^2)^{1/2}\,. \tag{7}$$

When both t_1 and t_2 are large and positive, there is a saddle point solution that connects them fairly directly, but this contributes to the matrix element of $a^\dagger_{\text{out}} a_{\text{out}}$ because the upper and low limits in the integral (6) contribute with opposite signs. The piece that we need must come from a path that winds around the branch cut at $t = ia/b$, which gives an extra sign and so

$$S \to -\frac{b}{2}(t_1^2 + t_2^2) - \frac{a^2}{2b}\ln\frac{4b^2 t_1 t_2}{a^2} + \frac{i\pi a^2}{2b}\,. \tag{8}$$

Then e^{iS} gives the outgoing wavefunctions times the production amplitude $e^{-\pi a^2/2b}$.

For the more general case of

$$m(t) = (a^2 + b^2 t^2)^{\alpha/2} \tag{9}$$

things are trickier: the branch cut no longer gives a simple sign in general, so we don't get the desired outgoing wavefunctions. Working backwards from the answer, the result must be

$$S \approx -\int_{ia/b}^{t_2} dt\,(a^2+b^2t^2)^{\alpha/2} + \int_{ia/b}^{t_1} dt\,(a^2+b^2t^2)^{\alpha/2}\,, \tag{10}$$

where in the second equation I have taken advantage of the sign ambiguity in solving the constraint (5): the signs then have the correct dependence to give the pair-creation amplitude. The path fails to be a classical solution where the two pieces meet at the branch cut, but it's OK because the semiclassical approximation breaks down there. To get the imaginary part we integrate vertically from the nearest point on the real axis to the branch cut. In general

$$\mathrm{Im}\,S \approx -2i\int_{t_1}^{t_*} m(t)\,, \tag{11}$$

where the nearest singularity in $m(t)$ is at $t_* = t_1 + it_2$. This agrees with Gubser. Generically the length of the contour is of order $m/\dot{m}$ and so the imaginary part is of order $m^2/\dot{m}$.

There's a small path integral subtlety when dealing with excited states. Let's generalize to

$$(\partial_t^2 + m^2(t) - \partial_x^2)\psi(t,x) = 0\,, \tag{12}$$

adding an extra coordinate. In a momentum eigenstate this is just replacing $m^2 \to m^2 + k^2$. However, the Lagrangian is now

$$\frac{1}{2}\,(-\dot{t}^2 + \dot{x}^2 - m^2(t))\,, \tag{13}$$

and replacing $\dot{x} \to k$ gives the wrong sign: we have to flip it. This can be understood as resulting from folding into the initial and final wavefunctions e^{ikx}. We will also need

$$(\partial_t^2 + m^2(t) - \partial_x^2 + x^2)\psi(t,x) = 0\,. \tag{14}$$

Here, $-\partial_x^2 + x^2$ should be replaced by $2n+1$, where n is the number of oscillator excitations (and the zero point $+1$ will be irrelevant for cosmic strings). Again to get this from the path integral we have to flip the sign of the kinetic terms. Evidently the prescription is to use the Lagrangian action to get the e.o.m., but them plug them into the integrated Hamiltonian to get the bounce action, except for the t coordinate which is kept in Lagrangian form because we are evaluating the path integral with fixed t endpoints.

2 Circular string production

The Polyakov action in conformal gauge is

$$S = \frac{1}{2}\int d\tau\,d\sigma\,\mu(t)(-\dot{t}^2 + t'^2 + \dot{X}^2 - X'^2)\,. \tag{15}$$

We consider a circular string with some center of mass motion (in a direction orthogonal to the plane of the string),

$$X = x(\tau) + r(\tau)(\hat{1}\cos\sigma + \hat{2}\sin\sigma)\,, \quad t = t(\tau)\,, \tag{16}$$

for which

$$S = \pi \int d\tau\, \mu(t)(-\dot{t}^2 + \dot{x}^2 + \dot{r}^2 - r^2)\,. \tag{17}$$

The equations of motion are

$$\dot{(\mu\dot{r})} = -\mu r\,, \quad \dot{(\mu\dot{x})} = 0\,, \quad \dot{t}^2 = \dot{x}^2 + \dot{r}^2 + r^2\,. \tag{18}$$

The final equation is from the constraint. For the center of mass motion we have $2\pi\mu\dot{x} = k$.

According to our prescription above, we then plug this into the action

$$I = \pi \int d\tau\, \mu(t)(-\dot{t}^2 - \dot{x}^2 - \dot{r}^2 - r^2) = -2\pi \int d\tau\, \mu(t)\dot{t}^2 = -2\pi \int dt\, \mu(t)\dot{t}\,. \tag{19}$$

The exponent B in the Bogoliubov coefficient is twice the imaginary part of this, integrated from the real axis to the branch cut.

2.1 Frozen approximation

Let us first check the case $\dot{r} \ll r$ against Eva's LL solution. We have $\dot{t}^2 = r^2 + k^2/4\pi^2\mu^2$, and so for $\mu^2 = a^2 + b^2t^2$ we have

$$B = 4\pi\,\mathrm{Im} \int_0^{t_*} dt\, (r^2a^2 + r^2b^2t^2 + k^2/4\pi^2)^{1/2} = \frac{\pi^2 r a^2}{b} + \frac{k^2}{4br}\,. \tag{20}$$

Note that the position of the branch cut depends on k.

2.2 Adiabatic approximation

Consider now the adiabatic approximation, where the change in tension is small during one period of the string, $\dot{\mu}/\mu \ll 1$ (note that the period is 2π in the world-sheet τ). In this limit we can solve the r equation via WKB,

$$r = \rho\mu^{-1/2}\cos\tau\,, \tag{21}$$

where ρ is a constant of the motion. In the same approximation,

$$\dot{t}^2 = \frac{\rho^2}{\mu} + \frac{k^2}{4\pi^2\mu^2}\,. \tag{22}$$

The mass of the string state is then $2\pi\rho\mu^{1/2} = (4\pi\mu[N+\tilde{N}])^{1/2}$, giving the right- plus left-moving excitation level $N+\tilde{N} = \pi\rho^2$.

The bounce action is now given by Eq. (19). To get an explicit expression let $\mu^2 = a^2+b^2t^2$ and set $k=0$, so

$$B = \frac{4\pi C\rho a^{3/2}}{b}\,, \quad C = \int_0^1 dx\,(1-x^2)^{1/4}\,. \tag{23}$$

More generally,

$$B = 4\pi\rho\,\mathrm{Im}\int_0^{t_*} dt\,\mu^{1/2} \sim \rho\mu^{3/2}/\partial_t\mu\,, \tag{24}$$

where t_* is the nearest branch point to the origin.

2.3 Matching

The condition for the adiabatic approximation to be valid is

$$\dot{\mu}/\mu = \rho\partial_t\mu/\mu^{3/2} < 1\,. \tag{25}$$

For excited strings this is stronger than both $\partial_t\mu/\mu^{3/2} < 1$ and $\partial_t m/m^2 \sim \partial_t\mu/\rho\mu^{3/2} < 1$. Moreover, it is exactly the same as the causality condition $\partial_t r < 1$.

For $\mu = (a^2+b^2t^2)^{1/2}$, the maximum of $\partial_t\mu/\mu^{3/2}$ is of order $ba^{-3/2}$, while for large times it is of order $b^{-1/2}t^{-3/2}$. For given ρ, the adiabatic condition (25) is always satisfied at large enough times. It is satisfied at all times only if $a^{3/2}b^{-1} > \rho$, in which case $B > \rho^2$.

The adiabatic approximation always holds at large times, but for large enough ρ it breaks down at small times and so we must match the two regimes: for a given adiabatic motion r freezes at a value $r(\rho)$ given by

$$r(\rho) = \rho\mu^{-1/2} \quad \text{when} \quad \rho\partial_t\mu/\mu^{3/2} = 1\,. \tag{26}$$

Generally this happens for $\mu \approx bt$, in which case $r(\rho) = \rho^{2/3}b^{-1/3}$. The branch cut t_* lies in the frozen part of the motion, so we can insert this value for r into the frozen result (20),

$$B = \frac{\pi^2\rho^{2/3}a^2}{b^{4/3}} + \frac{\pi k^2}{4b^{2/3}\rho^{2/3}}\,. \tag{27}$$

For large ρ this is an enhancement over the adiabatic result, as expected.

3 General string motion

3.1 Adiabatic approximation

The action (15) gives for X the equation of motion

$$(\mu\dot{X})^{\cdot} = (\mu X')'\,. \tag{28}$$

Let us start in the fully adiabatic regime. If μ depends only on τ then

$$X(\tau,\sigma) = \mu^{-1/2}(f(\sigma+\tau) + g(\sigma-\tau))\,. \tag{29}$$

The constraint equation for t becomes in the adiabatic limit

$$\dot{t}^2 - t'^2 = \frac{1}{\pi\mu}\int_0^{2\pi} d\sigma\,(f'^2(\sigma) + g'^2(\sigma)) = \frac{N+\tilde{N}}{\pi\mu} + \frac{k^2}{4\pi^2\mu^2}\,. \tag{30}$$

This is consistent with t being independent of σ, this can be enforced using the residual gauge invariance. Note that the constraint from $T_{\tau\sigma}$ gives $f'^2(\sigma) = g'^2(\sigma) = N/2\pi = \tilde{N}/2\pi$. If the adiabatic condition is satisfied at all times, we can simply insert this result for $\dot{t}$ into the action,

$$B = 4\pi\,\mathrm{Im}\int_0^{t_*} dt\,\mu(t)\dot{t}\,, \tag{31}$$

and obtain the same result as for a circular string with $\rho^2 \to (N+\tilde{N})/\pi$.

Of course, the circular loop contracts to point and will presumable annihilate there, but this happens after the adiabatic and matching phases so presumably does not affect the production rate.

3.2 Nonadiabaticity

The complication now is that the adiabatic condition breaks down at different times for different wavelength along the string. Consider a mode $X_n \sim \mu^{-1/2}e^{in\sigma}h_n(\tau)$. The equation of motion (28) becomes

$$\ddot{h}_n = \left(\frac{\ddot{\mu}}{2\mu} - \frac{\dot{\mu}^2}{4\mu^2} - n^2\right)h_n\,. \tag{32}$$

The adiabatic solution $h_n = f_n e^{in\tau} + g_n e^{-in\tau}$ breaks down when $\dot{\mu}/\mu > |n|$, so for the longest wavelengths first. In the opposite regime, where the σ-derivative term is negligible, we see immediately the solutions $X_n =$ constant and $\dot{X}_n = 1/\mu$. I assume that we want the former, as the latter blows up towards $t = 0$, but this isn't totally clear because the center of mass motion is of the second type. The matching for the nth mode occurs at τ_n such that

$$\dot{\mu}(t_n)/\mu(t_n) = n\,. \tag{33}$$

Thus

$$X_n(\tau) = \begin{cases} X_n^0\,, & \tau < \tau_n\,, \\ X_n^0[\mu(t_n)/\mu(t)]^{1/2}\cos n(\tau-\tau_n)\,, & \tau > \tau_n\,. \end{cases} \tag{34}$$

Notice that g_n and f_n are both determined in terms of X_n, as a result of setting the $\dot{X}_n = 1/\mu$ mode to zero. I will write things in terms of $\phi_n = f_n e^{-in(\tau-\tau_n)}$ since this characterizes the asymptotic state,

$$f_n = \phi_n e^{in(\tau-\tau_n)}\,, \quad g_n(\tau) = \phi_n e^{-in(\tau-\tau_n)}\,, \quad X^0_n = 2\mu^{-1/2}(t_n)\phi_n\,. \tag{35}$$

The constraint determining $\dot{t}$ is local in σ, but I think that in most cases t will be nearly a function of τ, so I take the global constraint

$$\begin{aligned} \dot{t}^2 &= \frac{1}{2\pi}\int d\sigma\,(\dot{X}^2 - X'^2) \\ &= \sum_{n:\tau_n<\tau}\frac{4n^2}{\mu(t_n)}|\phi_n|^2 + \sum_{n:\tau_n>\tau}\frac{4n^2}{\mu(t)}|\phi_n|^2\,. \end{aligned} \tag{36}$$

This expression is subtle, since the t_n on the right depend on integrating $\dot{t}$ to get $t(\tau)$. This is then inserted into the bounce acton (31). As expected the freezing ($t \to t_n$ in the first term) increase the rate over the adiabatic result. I think that (possibly with some of the assumptions clarified) this approach should make the causality manifest. What I want to do now is to think about summing this against the density of states to see which which term in (36) dominates, e.g. if the string has a lot of short wiggles then the second term might dominate and we'd revert to the adiabatic result. It seems to me that if $a^{3/2} < b$ then we're necessarily in a Hagedorn phase, since the adiabatic result is already dominated by the density of states.

4 Black hole case

4.1 setup, generalities

Let us consider the possibility of string production along the horizon of a black hole, where the Schwarzchild "t" coordinate shrinks to zero. In the t and r directions, the string sigma model effectively has a tension that goes to zero at the horizon. To see this, write the BH metric in conformal coordinates in those directions:

$$ds^2 = (\frac{r_0}{r} - 1)(-d\rho^2 + dt^2) + r^2 d\Omega^2 \tag{37}$$

with

$$\frac{d\rho}{dr} = \frac{1}{\frac{r_0}{r} - 1}, \qquad \rho = r + r_0\log(r_0 - r) + const \tag{38}$$

If we focus on the ρ (time) and t (one of the space) dimensions, the sigma model

$$S_{worldsheet} = \frac{1}{\alpha'}\int d\tau d\sigma G_{MN}\partial X^M \partial X^N = \int d\tau d\sigma \left\{(\frac{r_0}{r} - 1)(-\partial\rho^2 + \partial t^2) + r(\rho)^2\partial\Omega^2\right\} \tag{39}$$

has a prefactor which behaves like an effective tension,

$$\mu = \frac{r_0}{r} - 1 \tag{40}$$

From this we can form the figures of merit noted above. *Please see the next subsection 4.2 below for the continuation of the direct calculation.* For example, if we look for the analogue of the frozen approximation, with the string stretched in the t direction, the worldsheet constraint gives

$$(\frac{d\rho}{d\tau})^2 = t_{str}^2 \tag{41}$$

which gives

$$\frac{d\mu/d\tau}{\mu} \sim \frac{r_0 t_{str}}{r^2} \tag{42}$$

As in the above problem, this is large for large spatial extent t_{str}. In this frozen approximation regime we have

$$\frac{d\mu/d\rho}{\mu^2 t_{str}} \sim \frac{r_0 \alpha'}{t_{str} r(r_0 - r)} \tag{43}$$

which as expected is large close to the horizon.

One obvious obstruction to taking this seriously is the similarity to Minkowski space in a Milne slicing, with metric

$$ds^2 = e^{2\rho_m/R}(-d\rho_m^2 + dy^2) + dx_\perp^2 \tag{44}$$

for which the corresponding formulas are

$$\mu_m = e^{2\rho_m/R} \tag{45}$$

$$\frac{d\mu_m/d\tau}{\mu_m} \sim \frac{y_{str}}{R} \tag{46}$$

and

$$\frac{d\mu_m/d\rho_m}{\mu_m^2 y_{str}} \sim \frac{\alpha' e^{-2\rho_m/R}}{y_{str} R} \tag{47}$$

Of course there shouldn't be production in flat space, and the two look similar at the level of these diagnostics. These results are appropriate to a certain definition of positive frequency discussed below, a standard one in the particle case. In a black hole, the horizon is globally different from an artificial Milne horizon in flat spacetime, so the answers will not be identical in any case.[1] Since Bogoliubov transformations between modes arise in various places in black hole physics, it would be good to understand their generalization regardless.

[1]Also, there are other effects, such as the matter forming the black hole on which strings may end.

4.2 Contour integration calculation: Minkowski

4.2.1 particle case

Consider the Milne metric

$$ds^2 = -dT^2 + T^2 dy^2 + dx_\perp^2 \tag{48}$$

Let's start with the particle case. Imposing the constraint and the other wordline equations of motion gives us

$$I = \int dT\dot{T} = \int dT \sqrt{k_\perp^2 + m^2 + \frac{\ell_y^2}{T^2}} \tag{49}$$

(Note that y and k_y are dimensionless here.) Integrating around the branch cut at $T_* = ik_y/\sqrt{k_\perp^2 + m^2}$ gives us

$$B \sim \ell_y \tag{50}$$

This agrees with the standard estimate $B \sim \omega^2/(d\omega/dT)$ where the latter is evaluated at a T for which the physical momentum ℓ_y/T is of order $\sqrt{k_\perp^2 + m^2}$; for later T this quantity is suppressed since the time dependence in the $p_y = \ell_y/T$ modes becomes unimportant.

This agrees with [1], equation 27, for the overlap between Minkowski and Milne modes (in one of two ways they discuss of organizing the Milne modes, one they ascribe to Sommerfield, from [2]). The claim is that in this decomposition the Minkowski vacuum looks like a big bang in Milne, with a thermal spectrum. That is,

$$|\beta|^2 \sim e^{-k_y} = e^{-p_y T} \tag{51}$$

(where again T is our Milne time coordinate (48)). The temperature formally goes to infinity as $T \to 0$ (the far past).

This is formally similar to Rindler, maybe there is an analogy there but the interpretation must be somewhat different. In both cases we are (in this subsection) talking about the Minkowski vacuum.

It is clear technically why we land on this definition of positive frequency discussed in the Sommerfield paper [2] here, since we have an action going like $\int \ell_y dT/T = \ell_y \log(T) = \ell_y \alpha$ (α being Sommerfield's notation). Presumably we could adopt the calculation to pick the other notion of positive frequency he and others discuss in which positive frequency in Minkowski can be expanded entirely in terms of positive frequency in Milne.

4.2.2 string case

For the string case, imposing the constraint and the equations of motion on the worldsheet as above, here for a string stretched in the y direction, we get

$$I = \int dT\dot{T} = \int dT \sqrt{k_\perp^2 + T^2 y^2 + k_y^2/T^2} \tag{52}$$

and we are interested in B which is the imaginary part of this action, with the integral done over an appropriate contour. For simplicity consider for example $k_y = 0$. Then the action is

$$B = 4\pi Im \int_0^{T_*} dT \sqrt{k_\perp^2 + y^2 T^2} \sim \frac{k_\perp^2}{y} \tag{53}$$

In the last step we used a contour going around the branch cut at $T_* = ik_\perp/y$. In particular, our action here is precisely of the form (20), but with $a = 0$ (we set $\alpha' \sim 1$ here).

4.3 Contour integration calculation: BH

Let's do the string and the particle cases and compare. In a black hole with metric

$$ds^2 = -\frac{dr^2}{(\frac{r_0}{r} - 1)} + (\frac{r_0}{r} - 1)dt^2 + r^2 d\Omega^2 \tag{54}$$

we have a sigma model for a string stretched along the t direction of the form (no oscillators yet included)

$$S \sim \int d\tau \left\{ \frac{-\dot{r}^2}{(\frac{r_0}{r} - 1)} + (\frac{r_0}{r} - 1)\dot{t}^2 + r^2\dot{\Omega}^2 + (\frac{r_0}{r} - 1)t^2 \right\} \tag{55}$$

whereas for a particle we have

$$\int d\tau \left\{ -\dot{r}^2 (\frac{r_0}{r} - 1) + (\frac{r_0}{r} - 1)\dot{t}^2 + r^2\dot{\Omega}^2 + m^2 \right\} \tag{56}$$

Plugging in the solution of the constraint in each case gives

$$B = Im \int \frac{dr}{\sqrt{\frac{r_0}{r} - 1}} \sqrt{\frac{k_t^2}{\frac{r_0}{r} - 1} + \frac{\ell_\Omega^2}{r^2} + (\frac{r_0}{r} - 1)t^2 + osc} \tag{57}$$

for the string, and

$$B = Im \int \frac{dr}{\sqrt{\frac{r_0}{r} - 1}} \sqrt{\frac{k_t^2}{\frac{r_0}{r} - 1} + \frac{\ell_\Omega^2}{r^2} + m^2} \tag{58}$$

for the particle, where ℓ_Ω is the angular momentum quantum number in the sphere directions, and k_t is the momentum quantum number in the (spatial) t direction.

It is straightforward to analyze these in a similar way to the above, determining the branch cuts or other singularities and integrating around the nearest one to the real axis.

For the particle, one simple case is to take $m = \ell_\Omega = 0$, which gives

$$B \sim Im \int dr \frac{k_t}{\frac{r_0}{r} - 1} \sim 2\pi r_0 k_t \sim k_t/H \tag{59}$$

Where in this case, we integrated around the pole at $r = r_0$ and used the the Hubble scale H in the interior "cosmology" scales like $1/r_0$. This result is very reasonable, consistent with the normal adiabatic theorem/intuition. As a more basic sanity check, if we turn on only the (time-independent) mass m, we should get no contribution. This works out: then there is only one branch point at $r = r_0$, so we don't have an analogous imaginary contribution to the action.

Next, for the string, for example in the simplifying case $k_t = 0$, there is a branch cut that starts at $r = r_0$ and ends at $r_* = (1/2)(r_0 + \sqrt{r_0^2 + 4\ell_\Omega^2/t^2})$. Integrating around this gives a result

$$B \sim \frac{\ell_\Omega^2}{(t/\alpha')r_0} \sim \frac{p_\Omega^2}{(1/r_0)(t/\alpha')} \tag{60}$$

where p_Ω is the physical momentum corresponding to the angular momentum ℓ_Ω in the sphere directions. This result is very similar to the result (20) in the original calculation, but without the bare tension "a" term. The second form here also makes clear that the role of the time dependent tension "b" in the original problem is here being played by the scale $1/r_0$, which is the scale of the BH temperature, equivalently the Hubble scale in the interior "cosmology".

4.3.1 Previous description (a little more wordy in different coords)

Let's just try to do it. Here we follow our nose and imitate what we did above in section 2, now in the black hole case. Our time coordinate is ρ (or equivalently r), and the string is stretched in the spatial t direction, with some momentum k in the sphere direction. We have the worldsheet action

$$\begin{aligned} S_{worldsheet} &= \frac{1}{\alpha'}\int d\tau d\sigma G_{MN}\partial X^M \partial X^N = \int d\tau d\sigma \left\{ (\frac{r_0}{r} - 1)(-\partial\rho^2 + \partial t^2) + r(\rho)^2 \partial\Omega^2 \right\} \\ &= \int d\tau \mu(\rho) \left\{ -\dot\rho^2 + \dot t_{str}^2 - t_{str}^2 + \frac{r(\rho)^2}{\mu(\rho)}\dot\Omega^2 \right\} \end{aligned} \tag{61}$$

The constraint from this is (as in (18))

$$\dot\rho^2 = \dot t_{str}^2 + t_{str}^2 + \frac{r(\rho)^2}{\mu(\rho)}\dot\Omega^2 \tag{62}$$

and the momentum k is given by (slightly schematically, Ω is one of the shrinking spatial S^2 coordinates transverse to the stretching coordinate t)

$$\dot{(r(\rho)^2\dot\Omega)} = 0 \quad \Rightarrow \quad r(\rho)^2\dot\Omega = k \tag{63}$$

Plugging this into the analogue of (19) gives us

$$I = -2\pi \int d\rho \mu(\rho)\dot{\rho} = -2\pi \int dr \frac{d\rho}{dr} \dot{\rho}\mu \tag{64}$$

As above, we work in the regime $\dot{t} \ll t$, and we can write this in terms of r. Doing this we get exponent will be

$$B = 4\pi Im \int dr \frac{1}{\sqrt{\frac{r_0}{r} - 1}} \sqrt{\frac{k^2}{r^2} + t_{str}^2 (\frac{r_0}{r} - 1)} \tag{65}$$

where we have to integrate over some appropriate contour.

Here, we follow sections 1 and 2 above (and earlier refs) and use a prescription in which we integrate in r up to where the effective mass vanishes, a place where the action is imaginary.

$$\sqrt{\frac{k^2}{r^2} + t_{str}^2 (\frac{r_0}{r} - 1)} \quad |_{r_*} \equiv 0 \Rightarrow r_{*\pm} = (1/2)(r_0 \pm \sqrt{r_0^2 + 4\frac{k^2}{t_{str}^2}}) \tag{66}$$

These values of r are not in the range between 0 and r_0 which is inside the BH. The two values are either outside the horizon (r_{*+}) or past the singularity (r_{*-}). Let's consider the former for now but maybe both should be considered.

This gives us a nice imaginary part of the action, with a form generally expected from the previous figures of merit, as follows. Our action was

$$B = 4\pi \mathrm{Im} \int_{\mathrm{r}_0}^{\mathrm{r}_{*+}} \mathrm{dr} \frac{1}{\sqrt{\frac{\mathrm{r}_0}{\mathrm{r}} - 1}} \sqrt{\mathrm{t}_{\mathrm{str}}^2 (\frac{\mathrm{r}_0}{\mathrm{r}} - 1) + \frac{\mathrm{k}^2}{\mathrm{r}^2}} \tag{67}$$

$$= 4\pi \int_{r_0}^{r_{*+}} dr \frac{1}{\sqrt{1 - \frac{r_0}{r}}} \sqrt{-t_{str}^2 (1 - \frac{r_0}{r}) + \frac{k^2}{r^2}} \tag{68}$$

$$= 4\pi \int_{r_0}^{r_{*+}} dr \sqrt{\frac{k^2}{r(r - r_0)} - t_{str}^2} \tag{69}$$

Let's work in the regime $k^2 \ll t_{str}^2 r_0^2$, so that

$$r_{*+} \approx r_0 + \frac{k^2}{(r_0 t_{str}^2)}. \tag{70}$$

In this limit the integral simplifies, giving

$$B = 4\pi \int_{r_0}^{r_{*+}} dr \sqrt{\frac{k^2}{r_0(r - r_0)} - t_{str}^2} = 2 \int_0^{\frac{k}{t_{str}\sqrt{r_0}}} du \sqrt{\frac{k^2}{r_0} - t_{str}^2 u^2} = \frac{2\pi^2 k^2}{r_0 t_{str}} \tag{71}$$

This result seems pretty analogous to the answer (20), specifically the second term. Recall that here we have a string stretched in the t direction, so our t_{str} here is analogous to "r" in the circular string calculation. And note that we do not have an analogue of the first term because the effective tension $\mu(\rho)$ in our case goes through zero at the horizon. As in the above case, this is pair production, and the members of the pair will quickly annihilate into radiation. But we probably need to redo this in the infalling frame to get an appropriate definition of positive frequency here.

Note that in sections 1 and 2 we used a bit of answer analysis to work out the contour in more general cases than the Landau/Liftshitz example. In the present context, the "obvious" answer is no firewall, but there are arguments that this is wrong for the BH case.

References

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